Lectures on
Integrable equations of Benjamin–Ono type

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INTRODUCTION

Integrable PDEs

Since the late sixties, a number of Hamiltonian evolution PDEs were found to satisfy additional properties, implying infinitely many conservation laws, and which are expressed as a Lax pair identity,

$$\frac{dL}{dt} = [B, L] := BL - LB \ ,$$

where L, B are — usually differential — operators on an auxiliary Hilbert space $\mathcal{H}$. This identity often leads to a strategy for calculating explicitly the solution in terms of the initial data, via inverse spectral theory.

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The most famous example : the Korteweg–de Vries equation (Gardner–Green–Kruskal–Miura, 1967; Lax, 1968),

$$\partial_t u + 3\partial_x(u^2) = \partial_x^3 u \ , \ \mathcal{H} = L^2(\mathbb{R}) \ ,$$
$$L_u(f) := -\partial_x^2 f + uf \ , \ B_u f := 4\partial_x^3 f - 3u\partial_x f - 3\partial_x(uf) \ .$$

Why study such rare objects as integrable equations ?

- Because they provide specific powerful tools which allow us to establish results which are inaccessible otherwise. Typically : long time behaviour of solutions and small dispersion limit of KdV, modified scattering for defocusing cubic NLS (with no smallness assumption), global wellposedness of solutions to derivative NLS, ...
- Because perturbations of them keep parts of these features (Kolmogorov–Arnold–Moser, Nekhoroshev)

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However, in the above examples, these powerful tools (inverse spectral transform) are very heavy... and many people prefer not to use them and to use more standard PDE methods !

In these lectures, I would like to introduce a class of integrable equations where the integrable tools are easier , and the corresponding results are more general and more accessible.

The Benjamin–Ono equation

This equation was introduced in the late sixties (Benjamin, 1967; Davis–Acrivos, 1967) in order to model long, one-way internal gravity waves in a two-layer fluid with infinite depth, and reads

$$(BO) \qquad \partial_t u + \partial_x(u^2) = \partial_x |D| u \ .$$

Here $u = u(t, x)$ is a real valued function and $|D|$ denotes the Fourier multiplier associated to the symbol $|\xi|$ acting on functions on the real line.

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For this equation, there is a Lax pair structure too, but a new feature : the operators B, L are non local operators and act naturally on the Hardy space of holomorphic functions in the upper half–plane.

In these lectures, we shall see how to take advantage of this structure to derive a much simpler explicit formula for the solutions in terms of the initial data, bypassing some heavy inverse spectral theory. This will allow us to address various asymptotic regimes : long time, small dispersion.

The Calogero–Moser DNLS equation

$$i\partial_t v + \partial_x^2 v + \sigma |D|(|v|^2)v - \frac{1}{4}|v|^4 v = 0 \ , \ \sigma \in \{1, -1\} \ .$$

L^2–critical equation. Introduced in different contexts :

- Defocusing case $\sigma = -1$: special case of "intermediate" NLS equation (Pelinovsky, Grimshaw, 1995). Envelope for wave packets at the interface of two fluids with infinite depth.
- Focusing case $\sigma = 1$: Abanov, Bettelheim, Wiegmann, 2009. Formal continuum limit of the Calogero–Moser (1975) model posed on the real line

$$\frac{d^2 x_j}{dt^2} = \sum_{k \neq j} \frac{1}{(x_j - x_k)^3} \ .$$

Gauge transform and chirality condition

In the latter equation, we set

$$v(t,x) = \mathrm{e}^{-i\frac{\sigma}{2}\int_{-\infty}^{x}|u(t,y)|^2\,dy}u(t,x)\ ,$$

and we infer

$$(CMDNLS)\sigma \qquad i\partial_t u + \partial_x^2 u + \sigma(D + |D|)(|u|^2)u = 0\ .$$

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Observation : the “chirality” property

$$\mathrm{supp}(\hat{u}) \subset \mathbb{R}_+$$

(u belongs to the Hardy space) is formally preserved by the $(CMDNLS)_\sigma$ evolution.

From now on, we shall make this chirality assumption on our solutions.

Chiral (CMDNLS) as a mass critical version of (BO)

Introduce the Riesz–Szegő projector $\Pi := \mathbf{1}_{D\geq 0}$.
Then $(CMDNLS)_\sigma$ can be rewritten as

$(CMDNLS)\sigma$ $$i\partial_t u + \partial_x^2 u + 2\sigma \Pi D(|u|^2)u = 0 \ .$$

If u solves (BO), setting $w := \Pi u$, we get $u = w + \overline{w}$ and

(ΠBO) $$i\partial_t w - \partial_x^2 w - D(w^2 + 2\Pi(|w|^2)) = 0 \ .$$

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$$(\Pi BO) \qquad i\partial_t w - \partial_x^2 w - D(w^2 + 2\Pi(|w|^2)) = 0 \ .$$

It turns out that $(CMDNLS)_\sigma$ enjoys a Lax pair structure on the Hardy space, of the same type as the one of (BO).

Plan of the mini–course

These lectures are devoted to the study of the Benjamin–Ono equation (BO) and of the Calogero–moser DNLS equation $(CMDNLS)_\sigma$, both in the focusing case $(\sigma = 1)$ and the defocusing case $(\sigma = -1)$, with a special emphasis on the use of the Lax pair structures.

1. Wellposedness, Lax pair and conservation laws.
2. Explicit formulae ((BO) is better than (KdV) !!)
3. Solitons, multi–solitons, spectral theory and long time behaviour.
4. The small dispersion limit.

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Related results on the circle :
PG, T. Kappeler, P. Topalov (2018–23) for (BO),
Louise Gassot (2022–23) for small dispersion limit for (BO),
Rana Badreddine (2023) for $(CMDNLS)$ (=Calogero–Sutherland DNLS).

1A. LOCAL WELLPOSEDNESS

Local wellposedness for (BO)

Proposition

For any $R > 0$, there exists $T = T(R) > 0$ such that, for every $u_0 \in H^2_{\mathrm{real}}(\mathbb{R})$ with $\|u_0\|_{H^2} \leq R$, there exists a unique solution $u \in C([-T, T], H^2_{\mathrm{real}}(\mathbb{R})) \cap C^1([-T, T], L^2(\mathbb{R}))$ of the equation

$$\partial_t u - \partial_x |D_x| u + 2u\partial_x u = 0 \qquad (BO)$$

with $u(0) = u_0$. Furthermore, the map

$$u_0 \in H^2_{\mathrm{real}} \longmapsto u \in C([-T, T], H^2_{\mathrm{real}})$$

is continuous. If $u_0 \in H^s$ for some integer $s \geq 2$, then $u \in C([-T, T], H^s)$, and the flow map $u_0 \mapsto u(t)$ is continuous on H^s.

LWPBO, Proof

Kato's iterative scheme

$$\partial_t u^{n+1} - \partial_x |D_x| u^{n+1} + 2u^n \partial_x u^{n+1} = 0 \ , \ \ u^{n+1}(0) = u_0 \ .$$

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Lemma

Given $v_0 \in L^2$, $f \in L^1([-T, T], L^2)$, $u \in L^1([-T, T], \mathrm{Lip}_{\mathrm{real}})$, *there exists a unique* $v \in C([-T;, T], L^2)$ *such that*

$$\partial_t v - \partial_x |D_x| v + 2u\partial_x v = f \ , \ \ v(0) = v_0 \ .$$

Furthermore, for $t \in [-T, T]$, *the following estimate holds,*

$$\|v(t)\|_{L^2} \leq \|v_0\|_{L^2} + C \left| \int_0^t (\|\partial_x u(\tau)\|_{L^\infty} \|v(\tau)\|_{L^2} + \|f(\tau)\|_{L^2})\, d\tau \right| \ .$$

LWPBO, Proof, continued

Start with some $u^0 \in C([-T,T], H^2_{\mathrm{real}})$ such that $u^0(0) = u_0$. At each step n, the lemma provides $u^{n+1} \in C([-T,T], H^2_{\mathrm{real}})$ with, $\forall t \in [0,T]$,

$$\partial_t u^{n+1} - \partial_x |D_x| u^{n+1} + 2u^n \partial_x u^{n+1} = 0 \ , \qquad u^{n+1}(0) = u_0$$

$$\|u^{n+1}(t) - u^n(t)\|_{L^2} \leq C \int_0^t \|\partial_x u^n(\tau)\|_{L^\infty} \quad [\|u^{n+1}(\tau) - u^n(\tau)\|_{L^2} + \|u^n(\tau) - u^{n-1}(\tau)\|_{L^2}]\, d\tau$$

$$\|u^{n+1}(t)\|_{H^2} \leq \|u_0\|_{H^2} + C \int_0^t \quad [\|\partial_x u^n(\tau)\|_{L^\infty} \|u^{n+1}(\tau)\|_{H^2} + \|u^n(\tau)\|_{H^2} \|\partial_x u^{n+1}(\tau)\|_{L^\infty}]\, d\tau \ .$$

LWPBO, Proof, continued

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$$\|u^{n+1}(t)\|_{H^2} \le \|u_0\|_{H^2} + C \int_0^t \quad [\|\partial_x u^n(\tau)\|_{L^\infty} \|u^{n+1}(\tau)\|_{H^2} + \|u^n(\tau)\|_{H^2} \|\partial_x u^{n+1}(\tau)\|_{L^\infty}]\, d\tau \ .$$

If $\|u_0\|_{H^2} \le R$, choose $T>0$ so small that $R\,\mathrm{e}^{\tilde{C}TR} \le 2R$ and $\sup_{|t|\le T} \|u^0(t)\|_{H^2} \le 2R$. Then, by Grönwall's inequality,

$$\forall n \ge 0 \ , \ \sup_{|t|\le T} \|u^n(t)\|_{H^2} \le 2R \ , \ \sum_{n=0}^{\infty} \sup_{|t|\le T} \|u^{n+1}(t) - u^n(t)\|_{L^2} < +\infty \ .$$

LWPBO, Proof, conclusion

The L^2 contraction argument also leads to uniqueness of the solution in $C_w([-T,T],H^2_{\mathrm{real}}) \cap C([-T,T],L^2)$ and to continuity of the flow map.

The H^2 bound can be extended to H^s bound for $s > 2$ on the same time interval $[-T,T]$.

General considerations (Bona–Smith, Tao's frequency envelopes,...) lead to strong continuity $u \in C([-T,T],H^2_{\mathrm{real}})$. $\square$

Local wellposedness for $(CMDNLS)$

We work on the Hardy–Sobolev spaces

$$H^s_+(\mathbb{R}) = \{u \in H^s(\mathbb{R}) : \mathrm{supp}(\hat{u}) \subset \mathbb{R}_+\}$$

Proposition

For any $R > 0$, $\sigma \in \{1, -1\}$, there is some $T(R) > 0$ such that, for every $u_0 \in H^2_+(\mathbb{R})$ with $\|u_0\|_{H^2} \leq R$, there exists a unique solution $u \in C([-T, T]; H^2_+(\mathbb{R}))$ of

$$i\partial_t u + \partial_x^2 u + 2\sigma \Pi D(|u|^2)u = 0 \ , \qquad (CMDNLS)_\sigma$$

with $u(0) = u_0$. Furthermore, the H^s-regularity of u_0 for some integer $s \geq 2$ is propagated on the whole maximal interval of existence of u, and the flow map $u_0 \mapsto u(t)$ is continuous on H^s_+.

Main additional argument

Rewriting the equation

$$\partial_t u - i\partial_x^2 u - 2\sigma\Pi(\overline{u}\partial_x u)u = 2\sigma\Pi(u\partial_x\overline{u})u \qquad (CMDNLS)_\sigma$$

Lemma

If $u \in H_+^{\frac{3}{2}}(\mathbb{R})$*, then*

$$\|\Pi(u\partial_x\overline{f})\|_{L^2}^2 \leq \frac{1}{2\pi}(D\partial_x u, \partial_x u)\|f\|_{L^2}^2 \ .$$

If $u, v \in H_+^2(\mathbb{R})$*, then* $\|\Pi(u\partial_x\overline{v})\|_{H^2} \leq C\|u\|_{H^2}\|v\|_{H^2}$ *. with some constant* $C > 0$*.*

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If $u, v \in H_+^2(\mathbb{R})$*, then* $\|\Pi(u\partial_x\overline{v})\|_{H^2} \leq C\|u\|_{H^2}\|v\|_{H^2}$ *. with some constant* $C > 0$*.*

Kato's scheme is then

$$\partial_t u^{n+1} - i\partial_x^2 u^{n+1} - 2\sigma\Pi(\overline{u}^n\partial_x u^{n+1})u^n = 2\sigma\Pi(u^n\partial_x\overline{u}^n)u^n$$

Proof of the lemma

$$\widehat{\Pi(u\partial_x \overline{f})}(\xi) = -i\int_0^\infty \hat{u}(\xi+\eta)\eta\overline{\hat{f}(\eta)}\frac{d\eta}{2\pi} \quad \text{for} \quad \xi \geq 0.$$

$$\begin{aligned}
|\widehat{\Pi(u\partial_x \overline{f})}(\xi)|^2 &\leq \left|\int_0^\infty |\hat{u}(\xi+\eta)|\eta|\hat{f}(\eta)|\frac{d\eta}{2\pi}\right|^2 \\
&\leq \int_0^\infty |\hat{u}(\xi+\eta)|^2(\xi+\eta)^2\frac{d\eta}{2\pi}\cdot\int_0^\infty |\hat{f}(\eta)|^2\frac{d\eta}{2\pi} \\
\|\Pi(u\partial_x\overline{f}\|_{L^2}^2 &\leq \int_0^\infty\int_0^\infty |\hat{u}(\xi+\eta)|^2(\xi+\eta)^2\frac{d\eta}{2\pi}\frac{d\xi}{2\pi}\|f\|_{L^2}^2 \\
&\leq \int_0^\infty |\hat{u}(\zeta)|^2\zeta^3\frac{d\zeta}{4\pi^2}\|f\|_{L^2}^2 = \frac{1}{2\pi}(D\partial_x u, \partial_x u)\|f\|_{L^2}^2.
\end{aligned}$$

Second statement : first statement combined with Sobolev and identity

$$\partial_{xx}\Pi(u\partial_x\overline{v}) = \Pi[u\partial_x(\partial_{xx}\overline{v})] + 2\Pi(\partial_x u\partial_{xx}\overline{v}) + \Pi(\partial_{xx}u\partial_x\overline{v}).$$

1B. LAX PAIRS AND CONSERVATION LAWS

The Hardy space setting

$$L^2_+(\mathbb{R}) := \{f \in L^2(\mathbb{R}) : \forall \xi < 0, \hat{f}(\xi) = 0\}$$
$$= \left\{ f \text{ holomorphic on } \mathbb{C}_+ : \sup_{y>0} \int_{\mathbb{R}} |f(x+iy)|^2 \, dx < +\infty \right\}$$

Associated Riesz–Szegő projector $\widehat{\Pi f}(\xi) = \mathbf{1}_{\xi \geq 0} \hat{f}(\xi)$, or

$$\Pi f(z) = \frac{1}{2i\pi} \int_{\mathbb{R}} \frac{f(x)}{x-z} \, dx \ , \ z \in \mathbb{C}_+ \ .$$

Given $b \in L^\infty$, define the Toeplitz operator of symbol b,

$$T_b : L^2_+ \to L^2_+ \ , \ f \mapsto T_b f := \Pi(bf) \ .$$

Example : If

$$b(x) = \frac{1}{x-p} \quad , \quad T_b f(x) = \begin{cases} \frac{f(x)}{x-p} & \text{if } p \in \mathbb{C}_- \ , \\ \frac{f(x)-f(p)}{x-p} & \text{if } p \in \mathbb{C}_+ \ . \end{cases}$$

A crucial lemma

Lemma

For $a, b \in L^\infty, f \in L^2_+$,

$(T_{ab} - T_a T_b)f = \Pi\Big(\Pi(a)(\mathrm{Id} - \Pi)\{(\mathrm{Id} - \Pi)(b)f\}\Big)$

Proof. Main observation : if f, g have positive (resp. negative) frequencies, then fg has positive (resp.negative) frequencies.

$$T_{ab}f - T_a T_b f = \Pi(abf) - \Pi(a\Pi(bf)) = \Pi(aU), \quad U := (\mathrm{Id} - \Pi)(bf)$$

$$\Pi(aU) = \Pi(\Pi(a)U) + \Pi((\mathrm{Id} - \Pi)(a)U) = \Pi(\Pi(a)U).$$

Finally, write $bf = \Pi(b)f + (\mathrm{Id} - \Pi)(b)f$, and $\Pi(b)f \in L^2_+$, so that

$$U = (\mathrm{Id} - \Pi)(bf) = (\mathrm{Id} - \Pi)\{(\mathrm{Id} - \Pi)(b)f\} .$$

□

The Lax pair for (BO)

For $u \in H^2_{\mathrm{real}}(\mathbb{R})$, define, with $D := -i\partial_x$,

$$L_u := D - T_u \in \mathcal{L}(H^1_+, L^2_+) ,$$
$$B_u := i(T_{|D_x|u} - T_u^2) \in \mathcal{L}(H^k_+, H^k_+), k = 0, 1 .$$

Theorem

If $u \in C(\mathbb{R}, H^2_{\mathrm{real}})$ solves (BO), then

$$\frac{dL_{u(t)}}{dt} = [B_{u(t)}, L_{u(t)}] .$$

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Theorem

If $u \in C(\mathbb{R}, H^2_{\mathrm{real}})$ solves (BO), then

$$\frac{dL_{u(t)}}{dt} = [B_{u(t)}, L_{u(t)}] \ .$$

Proof. Observe that $T_u^* = T_u$ and $B_u^* = -B_u$. We have

$$\frac{d}{dt} L_{u(t)} = -T_{\partial_t u(t)} = -T_{\partial_x |D_x| u(t)} + 2T_{u(t)\partial_x u(t)} := (1)$$

Proof of the Lax pair identity, continued

Since $[\partial_x, T_b] = T_{\partial_x b}$ and $D_x = \frac{1}{i}\partial_x$,

$(1) = i\,[T_{|D_x|u}, D_x] + 2T_{u\partial_x u} = i\,[T_{|D_x|u}, D_x - T_u] + 2T_{u\partial_x u} + i[T_{|D_x|u}, T_u].$

Consequently, $\frac{d}{dt}L_{u(t)} = i\,[T_{|D_x|u}, L_u] + 2T_{u\partial_x u} + i[T_{|D_x|u}, T_u].$

$(2) := i[T_{|D_x|u}, T_u]f = i\Big(T_{|D_x|u}T_u - T_{u|D_x|u}\Big)f + i\Big(T_{u|D_x|u} - T_uT_{|D_x|u}f\Big).$

Apply the lemma with $a = |D_x|u, b = u$, then $a = u, b = |D_x|u$.

$$(2) = \quad -i\Pi\Big(\Pi(|D_x|u)(\mathrm{Id}-\Pi)\big\{(\mathrm{Id}-\Pi)(u)f\big\}\Big) + \\ i\Pi\Big(\Pi(u)(\mathrm{Id}-\Pi)\big\{(\mathrm{Id}-\Pi)(|D_x|u)f\big\}\Big).$$

Proof of the Lax pair identity, conclusion

Since
$\Pi(|D_x|v)(x) = \frac{1}{i}(\Pi\partial_x v)(x)$, $(\mathrm{Id} - \Pi)(|D_x|v)(x) = i(\mathrm{Id} - \Pi)\partial_x v(x)$,

$(2) = -\Pi\Big(\Pi(\partial_x u)(\mathrm{Id}-\Pi)\{(\mathrm{Id}-\Pi)(u)f\}\Big) - \Pi\Big(\Pi(u)(\mathrm{Id}-\Pi)\{(\mathrm{Id}-\Pi)(\partial_x u)f\}\Big)$.

Applying again the lemma, we obtain

$$\begin{aligned}(2) = i[T_{|D_x|u}, T_u]f &= -\Big(T_{u\partial_x u} - T_{\partial_x u}T_u\Big)f - \Big(T_{u\partial_x u} - T_u T_{\partial_x u}\Big)f,\\ &= -2T_{u\partial_x u}f + T_{\partial_x u}T_u f + T_u T_{\partial_x u} f.\end{aligned}$$

Then observe that

$$\begin{aligned}[T_u^2, D_x] &= T_u[T_u, D_x] + [T_u, D_x]T_u = -\frac{1}{i}T_u T_{\partial_x u} - \frac{1}{i}T_{\partial_x u}T_u,\\ &= i(T_u T_{\partial_x u} + T_{\partial_x u}T_u),\end{aligned}$$

so that $i[T_{|D_x|u}, T_u] = -2T_{u\partial_x u} - i[T_u^2, D_x] = -2T_{u\partial_x u} - i[T_u^2, L_u]$.
Finally,

$$\frac{d}{dt}L_u = i\,[T_{|D_x|u}, L_u] - i[T_u^2, L_u] = [B_u, L_u]\ .$$

The Lax pair for $(CMDNLS)_\sigma$

For $u \in H^2_+(\mathbb{R})$, $\sigma \in \{1, -1\}$ define

$$L_u^\sigma := D - \sigma T_u T_{\overline{u}} \ , \ B_u^\sigma := \sigma(T_u T_{\partial_x \overline{u}} - T_{\partial_x u} T_{\overline{u}}) + i(T_u T_{\overline{u}})^2 \ .$$

Similarly, one can prove

Theorem

If $u \in C(I, H^2_+(\mathbb{R}))$ satisfies $(CMDNLS)_\sigma$,

$$\frac{dL^\sigma_{u(t)}}{dt} = [B^\sigma_{u(t)}, L^\sigma_{u(t)}] \ .$$

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Proof : exercise !

Conservation laws

Theorem

1) If u is a $H^{\max(2,k/2)}$ *solution of* (BO)*, then*

$$E_k(u) := \langle L_u^k(\Pi u), \Pi u \rangle$$

is conserved.

2) If u is a $H^{\max(2,k/2)}$ *solution of* $(CMDNLS)_\sigma$*, then*

$$E_k^\sigma(u) := \langle (L_u^\sigma)^k u, u \rangle$$

is conserved.

Conservation laws, proof

1) The BO case. Applying the Lax pair identity $\frac{dL_{u(t)}}{dt} = [B_{u(t)}, L_{u(t)}]$ to χ_ε with $\chi_\varepsilon(x) := (1 - i\varepsilon x)^{-1}$, $\varepsilon > 0$, and make $\varepsilon \to 0$. We get

$$\partial_t \Pi u = B_u \Pi u + i L_u^2 \Pi u \ .$$

2) The CMDNLS case. One can check directly that, of u satisfies $(CMDLS)_\sigma$),

$$\partial_t u = B_u^\sigma u - i(L_u^\sigma)^2(u) \ .$$

Conservation laws, proof

1) The BO case. Applying the Lax pair identity $\frac{dL_{u(t)}}{dt} = [B_{u(t)}, L_{u(t)}]$ to χ_ε with $\chi_\varepsilon(x) := (1 - i\varepsilon x)^{-1}$, $\varepsilon > 0$, and make $\varepsilon \to 0$. We get

$$\partial_t \Pi u = B_u \Pi u + i L_u^2 \Pi u \ .$$

2) The CMDNLS case. One can check directly that, of u satisfies $(CMDLS)_\sigma$),

$$\partial_t u = B_u^\sigma u - i(L_u^\sigma)^2(u) \ .$$

Calculate for instance for (BO),

$$\begin{aligned}
&\frac{d}{dt} E_k(u(t)) \\
&= \langle L^k_{u(t)} \partial_t \Pi u(t), \Pi u(t)\rangle + \langle L^k_{u(t)} \Pi u(t), \partial_t \Pi u(t)\rangle + \langle \partial_t [L^k_{u(t)}] \Pi u(t), \Pi u(t)\rangle \\
&= \langle L^k_u (B_u \Pi u + i L_u^2 \Pi u), \Pi u\rangle + \langle L^k_u \Pi u, B_u \Pi u + i L_u^2 \Pi u\rangle \\
&+ \langle (B_u L^k_u - L^k_u B_u) \Pi u, \Pi u\rangle \\
&= 0 \ .
\end{aligned}$$

Global wellposedness

Corollary

1) The initial value problems for (BO) and for $(CMDNLS)_{-1}$ (defocusing) are globally wellposed on H^k for $k \geq 2$, with uniform bounds.

2) The initial value problem for $(CMDNLS)_1$ (focusing) is globally wellposed on H^k for $k \geq 2$ under the condition $\|u\|_{L^2}^2 \leq 2\pi - \delta$, $\delta > 0$, with uniform bounds.

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Remarks. In the case 2), the critical mass 2π is the mass of the soliton (see further). Furthermore, one can prove that there is no finite time blow up for solutions with this mass.
This mass condition has been recently proved to be optimal by Kim–Kim–Kwon, who constructed finite time blow up solutions with mass bigger but arbitrarily close to 2π, using Martel–Merle– Raphaël modulation theory (see also Hogan–Kowalski).

2. THE EXPLICIT FORMULA

More from the Hardy space toolbox

Consider the Lax–Beurling semigroup

$$S(\eta)f(x) := \mathrm{e}^{i\eta x} f(x) \ , \ \eta \geq 0 \ .$$

Infinitesimal generator X= multiplication by x. Define the adjoint operator X^*, so that

$$\begin{aligned} S(\eta)^* &= T_{\mathrm{e}^{-i\eta x}} = \mathrm{e}^{-i\eta X^*} \ , \ \eta \geq 0 \ , \\ \mathrm{Dom}(X^*) &= \{f \in L^2_+(\mathbb{R}) : \exists \lambda_f \in \mathbb{C} : Xf + \lambda_f \in L^2(\mathbb{R})\}, \\ X^* f(x) &= xf(x) + \lambda_f \ . \end{aligned}$$

Notice that, $f \in \mathrm{Dom}(X^*)$ iff $\hat{f} \in H^1(0,\infty)$, hence $\hat{f}$ is right continuous at 0, and we may define

$$I_+(f) := \hat{f}(0^+) = 2i\pi\lambda_f = \lim_{\varepsilon \to 0^+} \langle f|\chi_\varepsilon\rangle_{L^2} \ .$$

A variant of the Cauchy integral

In general, one can prove, for every $z \in \mathbb{C}_+$, for every $f \in L^2_+(\mathbb{R})$,

$$(X^* - z\mathrm{Id})^{-1} f(x) = \frac{f(x) - f(z)}{x - z} \ , \ f(z) = \frac{1}{2i\pi} I_+((X^* - z\mathrm{Id})^{-1} f) \ .$$

Indeed, the function $g_z : x \mapsto \frac{f(x)-f(z)}{x-z}$ satisfies $(X - z)g_z + f(z) = f$,
hence $g_z = (X^* - z)^{-1} f$ and $2i\pi f(z) = I_+(g_z)$.
Other proof by inverse Fourier transform :

$$\begin{aligned} f(z) &= \frac{1}{2\pi} \int_0^{+\infty} \mathrm{e}^{iz\xi} \hat{f}(\xi)\, d\xi = \frac{1}{2\pi} \int_0^{+\infty} \mathrm{e}^{iz\xi} \lim_{\varepsilon \to 0^+} \langle S(\xi)^* f, \chi_\varepsilon \rangle \, d\xi \\ &= \frac{1}{2\pi} \int_0^{+\infty} \mathrm{e}^{iz\xi} \lim_{\varepsilon \to 0^+} \langle \mathrm{e}^{-i\xi X^*} f, \chi_\varepsilon \rangle_{L^2} \, d\xi \\ &= \frac{1}{2i\pi} \lim_{\varepsilon \to 0^+} \langle (X^* - z)^{-1} f, \chi_\varepsilon \rangle_{L^2} \ . \end{aligned}$$

Commutator identities

Lemma

For every $b \in H^1(\mathbb{R})$, *for every* $f \in \mathrm{Dom}(X^*)$, *we have* $T_b f \in \mathrm{Dom}(X^*)$ *and*

$$X^* T_b f - T_b X^* f = \frac{i}{2\pi} I_+(f) \Pi b \ .$$

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Consequences:

$$[X^*, B_u] = -2L_u - i[X^*, L_u^2]$$
$$[X^*, B_u^\sigma] = 2L_u^\sigma + i[X^*, (L_u^\sigma)^2]$$

The explicit formula

Theorem

1) (PG, 2022) The solution $u \in C(\mathbb{R}, H^2_{\rm real}(\mathbb{R}))$ *of the Benjamin–Ono equation with* $u(0) = u_0$ *is given by* $u(t,x) = \Pi u(t,x) + \overline{\Pi u(t,x)}$ *with*

$$\forall z \in \mathbb{C}_+ \ , \ \Pi u(t,z) = \frac{1}{2i\pi} I_+[(X^* - 2tL_{u_0} - z{\rm Id})^{-1}\Pi u_0] \ .$$

2) (R. Killip–T. Laurens–M. Vișan, 2024) The solution $u \in C(\mathbb{R}, H^2_+(\mathbb{R}))$ *of* $(CMDNLS)_\sigma$ *with with* $u(0) = u_0$ *is given by*

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See also Xi Chen (L^2 data for (BO)) ,
Rana Badreddine (CSDNLS=CMDNLS on the circle).

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See also Xi Chen (L^2 data for (BO)) ,
Rana Badreddine (CSDNLS=CMDNLS on the circle). Let us prove the result for (BO). Similar proof for $(CMDNLS)_\sigma$.

Proof 1 : using the Lax pair

Proposition

Define the family of unitary operators $\{U(t)\}_{t\in\mathbb{R}}$ *on* $L^2_+(\mathbb{R})$ *by*

$$U'(t) = B_{u(t)}U(t)\ ,\ U(0) = \mathrm{Id}\ .$$

Then

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Proof.

$$\frac{d}{dt}U(t)^*L_{u(t)}U(t) = U(t)^*\left(-B_{u(t)}L_{u(t)} + \frac{d}{dt}L_{u(t)} + L_{u(t)}B_{u(t)}\right)U(t) = 0$$

□

Proof 2, sketch

Since $u(t,.)$ is real valued, $u(t,x) = \Pi u(t,x) + \overline{\Pi u(t,x)}$.
Apply the variant of the Cauchy formula

$$f(z) = \frac{1}{2i\pi} \lim_{\varepsilon \to 0^+} \langle (X^* - z\mathrm{Id})^{-1} f, \chi_\varepsilon \rangle_{L^2}$$

to $f := \Pi u(t,.)$, and let the unitary operator $U(t)^*$ act on both sides of this inner product,

$$\Pi u(t,z) = \lim_{\varepsilon \to 0^+} \langle (U(t)^* X^* U(t) - z\mathrm{Id})^{-1} U(t)^* \Pi u(t) | U(t)^* \chi_\varepsilon \rangle \ .$$

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Calculate the quantities $U(t)^* \Pi u(t), U(t)^* \chi_\varepsilon, U(t)^* X^* U(t)$ in termes of u_0 only, using $U'(t) = B_{u(t)} U(t)$ and the crucial identities

$$\partial_t \Pi u = B_u \Pi u + i L_u^2 \Pi u \ ,$$

$$B_u \chi_\varepsilon = -i L_u^2 \chi_\varepsilon + o(1) \ ,$$

$$[X^*, B_u] = -2L_u - i[X^*, L_u^2] \ .$$

3. SOLITONS, MULTI–SOLITONS AND LONG TIME BEHAVIOUR

Recall : the explicit formula

Theorem

1) The solution $u \in C(\mathbb{R}, H^2_{\mathrm{real}}(\mathbb{R}))$ *of the Benjamin–Ono equation with* $u(0) = u_0$ *is given by* $u(t,x) = \Pi u(t,x) + \overline{\Pi u(t,x)}$ *with*

$$\forall z \in \mathbb{C}_+ \ , \ \Pi u(t,z) = \frac{1}{2i\pi} I_+[(X^* - 2tL_{u_0} - z\mathrm{Id})^{-1}\Pi u_0] \ .$$

2) The solution $u \in C(\mathbb{R}, H^2_+(\mathbb{R}))$ *of* $(CMDNLS)_\sigma$ *with with* $u(0) = u_0$ *is given by*

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Goal : use this formula for describing special solutions such that Πu_0 (resp. u_0) belongs to a finite dimensional vector space preserved by the actions of X^* and of L_{u_0} (resp. L^σ_u).

A spectral characterization theorem

Theorem

Let $N \geq 1$ be an integer.
1) If $u \in H^2_{\mathrm{real}}(\mathbb{R})$, there exists a N–dimensional subspace $\mathcal{E}$ of $Dom(X^*) \cap H^1_+$ such that $\Pi u \in \mathcal{E}$, $X^*(\mathcal{E}) \subset \mathcal{E}$ and $L_u(\mathcal{E}) \subset \mathcal{E}$ if and only if

$$u(x) = \sum_{j=1}^{N} \frac{2\mathrm{Im} p_j}{|x + p_j|^2} \ , \ p_1, \dots, p_N \in \mathbb{C}_+ \ .$$

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2) If $u \in H^2_+(\mathbb{R})$, there exists a N–dimensional subspace $\mathcal{E}$ of Dom$(X^*) \cap H^1_+$ such that $u \in \mathcal{E}$, $X^*(\mathcal{E}) \subset \mathcal{E}$ and $L_u^+(\mathcal{E}) \subset \mathcal{E}$ if and only if there exist $P, Q \in \mathbb{C}[x]$ with $\mathrm{deg} Q = N, \mathrm{deg P} \leq N - 1$, $Q^{-1}(0) \subset \mathbb{C}_-$, such that

$$u(x) = \frac{P(x)}{Q(x)} \quad \text{and} \quad P\overline{P} = i(Q'\overline{Q} - \overline{Q}'Q).$$

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$$u(x) = \frac{P(x)}{Q(x)} \quad \text{and} \quad P\overline{P} = i(Q'\overline{Q} - \overline{Q}' Q).$$

3) For every $u \in H^2_+(\mathbb{R})$, L_u^- has no eigenvalue.

The special case of solitons

In the previous theorem, the case $N = 1$ corresponds to soliton solutions,
1) In the (BO) case, (Amick–Toland 1991),

$$u(t,x) = \frac{2\mathrm{Im}(p)}{|x + p - \frac{t}{\mathrm{Im}(p)}|^2} \ , \ p \in \mathbb{C}_+ \ .$$

2) In the $(CMDNLS)_+$ case (PG–Lenzmann 2022), stationary waves

$$u(t,x) = \mathrm{e}^{i\theta} \frac{\sqrt{2\mathrm{Im}(p)}}{x + p} \ , \ p \in \mathbb{C}_+ \ .$$

More generally, traveling solitary waves are given by applying a Galilean transformation,

$$u(t,x) = \mathrm{e}^{i\theta - i\eta^2 t + i\eta x} \frac{\sqrt{2\mathrm{Im}(p)}}{x - 2\eta t + p} \ , \ p \in \mathbb{C}_+ \ , \ \eta \geq 0 \ .$$

Definition of an N–soliton

Definition

We say that u is a N–soliton for (BO) (resp. $(CMDNLS)_+$) if it satisfies the property 1) (resp. 2)) of the spectral characterisation theorem.

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N –solitons for (BO) (resp. $(CMDNLS)_+$) are invariant by the flow map of (BO) (resp. $(CMDNLS)_+$). Indeed, for instance for (BO),

$$U(t)\mathcal{E}_{pp}(L_{u_0}) = \mathcal{E}_{pp}(L_{u(t)}) \ ,$$

$$U(t)^*X^*U(t) = -2tL_{u_0} + \mathrm{e}^{itL_{u_0}^2}X^*\mathrm{e}^{-itL_{u_0}^2} \ .$$

$$U(t)^*\Pi u(t) = \mathrm{e}^{itL_{u_0}^2}\Pi u_0 \ .$$

Proof of the theorem

Lemma (Lax, 1959)

The non trivial closed subspaces M of $L^2_+(\mathbb{R})$ invariant by the semi–group $(S(\eta))_{\eta\geq 0}$ are exactly of the form

$$M = \theta L^2_+(\mathbb{R}) \ ,$$

where $\theta \in L^\infty_+(\mathbb{R})$ with $|\theta(x)| = 1$ a.e. on the real line ("inner function"). The special case $\dim(M^\perp) = N$ corresponds to

$$\theta(x) = \prod_{j=1}^{N} \frac{x+\overline{p}_j}{x+p_j} \ , \ p_1 \dots , p_N \in \mathbb{C}_+ \ .$$

Proof of the theorem, continued

In the (BO) case, just write that the space $\mathcal{E}$ must be

$$\mathcal{E} = (\theta L^2_+(\mathbb{R}))^\perp = \frac{\mathbb{C}_{N-1}[x]}{Q(x)} \ , \ Q(x) := \prod_{j=1}^{N} (x + p_j) \ ,$$

and that Πu belongs to this space. The condition on u reads $D\theta = \theta u$ and follows from the fact that $L_u = D - T_u$ satisfies

$$L_u \left(\frac{\mathbb{C}_{N-1}[x]}{Q(x)} \right) \subset \frac{\mathbb{C}_{N-1}[x]}{Q(x)} \ .$$

Similar proof for $(CMDNLS)_+$ with $L_u^+ = D - T_u T_{\overline{u}}$, $D\theta = \theta |u|^2$ and

$$u(x) = \frac{P(x)}{Q(x)} \ , \ Q(x) = \prod_{j=1}^{N} (x + p_j) \ .$$

Spectral theory of the Lax operators

Theorem

1) The (BO) case (Y. Wu, R. Sun). Let $u \in L^2_{\mathrm{real}}(\mathbb{R}, (1+x^2)dx)$. *Then* L_u *has only simple eigenvalues* λ_j, *with eigenfunctions satisfying*

$$-2\pi\lambda_j \|\varphi_j\|^2_{L^2} = |\langle \Pi u, \varphi_j\rangle_{L^2}|^2 \ , \ -\lambda_j I_+(\varphi_j) = \langle \varphi_j, \Pi u\rangle \ .$$

If u *is a* N*–soliton,* $\|\Pi u\|^2_{L^2} = 2\pi \sum_{j=1}^{N} |\lambda_j| \ .$

2) The focusing (CMDNLS) case (PG, E. Lenzmann). Let $u \in L^2_+(\mathbb{R})$. *Then* L_u^+ *has only simple eigenvalues, with eigenfunctions* φ_j *satisfying*

$$2\pi \|\varphi_j\|^2_{L^2} = |\langle u, \varphi_j\rangle_{L^2}|^2 \ .$$

If u *is a* N*–soliton, one of these eigenvalues is* 0, *with eigenvector* $1-\theta$, *and* $\|u\|^2_{L^2} = 2\pi N \ .$

Multisoliton solutions of $(CMDNLS)_+$ are global

Lemma

Assume u is a N–soliton for $(CMDNLS)_+$. For every $t \in \mathbb{R}$, $X^* + 2tL_u^+$ has no real eigenvalue on $\mathcal{E}_{pp}(L_u^+)$.

Let $\psi \in \mathcal{E}_{pp}(L_u^+)$ such that $(X^* + 2tL_u^+ - \mu)\psi = 0$ with $\mu \in \mathbb{R}$.
Imaginary part of the inner product with ψ implies
$I_+(\psi) = 0 = \langle \psi, 1 - \theta \rangle$, hence $\langle \psi, \varphi_1 \rangle = 0$.
We infer that $\psi = L_u^+ f$ for some $f \in \mathcal{E}_{pp}(L_u^+)$ with $\langle f, \varphi_1 \rangle = 0$. Then

$$L_u^+(X^* f + 2tL_u^+ f - \mu f) = -i \left(f - \frac{1}{2\pi} \langle f, u \rangle u \right) .$$

Inner product with φ_1 implies $\langle f, u \rangle = 0$.
Inner product with f implies $f = 0$. □

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Inner product with φ_1 implies $\langle f, u \rangle = 0$.
Inner product with f implies $f = 0$. $\square$
If u_0 is a N–soliton, the explicit formula implies that the rational function $u(t, .)$ does not blow up in finite time.

Description of the multisoliton solutions of (BO)

Theorem (Y. Matsuno (1984), R. Sun (2021))

Let $(\varphi_1, \dots, \varphi_N)$ *be an orthonormal basis of eigenfunctions for* L_{u_0} *with* $L_{u_0}\varphi_j = \lambda_j \varphi_j$ *and* $\langle \Pi u, \varphi_j \rangle > 0$. *Consider the* $N \times N$ *matrix* $\mathcal{M}$ *defined by*

$$\mathcal{M}_{jj} = \mathrm{Re}\langle X^* \varphi_j, \varphi_j \rangle - \frac{i}{2|\lambda_j|} \ , \ \mathcal{M}_{jk} = \frac{i}{\lambda_j - \lambda_k} \ , j \neq k$$

. *The solution* u *of* (BO) *with* $u(0) = u_0$ *is given by*

$$\begin{aligned} \Pi u(t,z) &= -i\langle (\mathcal{M} - 2t\mathrm{diag}(\lambda_1, \dots, \lambda_N) - z)^{-1} A, B \rangle_{\mathbb{C}^N} \ , \\ A : &= B = (1, \dots, 1)^T \ . \end{aligned}$$

Description of the multisoliton solutions of $(CMDNLS)_+$

Theorem (PG, E. Lenzmann, 2022)

Let $(\varphi_1, \dots, \varphi_N)$ be an orthonormal basis of eigenfunctions for L_{u_0} with $L^+_{u_0}\varphi_j = \lambda_j \varphi_j, \lambda_1 = 0$, and $\langle u, \varphi_j\rangle > 0$. Consider the $N \times N$ matrix $\mathcal{M}^+$ defined by

$$\mathcal{M}^+_{jj} = \mathrm{Re}\langle X^* \varphi_j, \varphi_j\rangle - \frac{i|I_+(u_0)|^2}{8\pi^2}\delta_{j1}\ ,\ \mathcal{M}^+_{jk} = \frac{i}{\lambda_j - \lambda_k}\ , j \neq k\ .$$

The solution u of $(CMDNLS)_+$ with $u(0) = u_0$ is defined for every $t \in \mathbb{R}$ and is given by

$$\begin{aligned} u(t,z) &= \frac{I_+(u_0)}{2i\pi}\langle (\mathcal{M}^+ + 2t\mathrm{diag}(\lambda_1, \dots, \lambda_N) - z)^{-1} A, B\rangle_{\mathbb{C}^N}\ , \\ A: &= (1, \dots, 1)^T\ ,\ B := (1, 0, \dots, 0)^T\ . \end{aligned}$$

The long time behaviour for N–solitons of (BO)

Using the above description, we infer

Theorem (Y. Matsuno (1984), R. Sun (2021))

Let u_0 be a N–soliton for (BO). Then the solution u satisfies

$$\lim_{t\to\pm\infty}\int_{\mathbb{R}}\left|u(t,x)-\sum_{j=1}^{N}\frac{2\mathrm{Im}p_j^{\infty}}{|x+p_j^{\infty}-\frac{t}{\mathrm{Im}p_j^{\infty}}|^2}\right|^2 dx=0\ ,$$

where

$$p_j^{\infty}:=-\mathrm{Re}\langle X^*\varphi_j,\varphi_j\rangle+\frac{i}{2|\lambda_j|}\ .$$

The long time behaviour for N–solitons of $(CMDNLS)_+$

Theorem (PG, E. Lenzmann, 2022)

Let u_0 be a N–soliton for $(CMDNLS)_+$ with $N \geq 2$. Then, for every $s > 0$, there exists $c_s > 0$ such that

$$\|u(t)\|_{H^s} \sim c_s \, |t|^{2s}$$

as $t \to +\infty$.

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Main argument : by spectral perturbation theory, for $|t|$ large enough, the eigenvalues $z_1(t), \dots, z_N(t)$ of the matrix $\mathcal{M}^+ + 2t\mathrm{diag}(\lambda_1, \dots, \lambda_N)$ satisfy

$$\mathrm{Im} z_1(t) = -\rho + O(t^{-1}) \ , \ \mathrm{Im} z_k(t) = -\frac{\rho}{4\lambda_k^2 t^2} + O(t^{-3}) \ ,$$

where $\rho := \frac{|I_+(u_0)|^2}{8\pi^2} > 0$.

Perspectives

- Work in progress : E. Blackstone, L. Gassot, PG, P. Miller. Long time soliton resolution for (BO) ?

$$u(t,x)=\sum_{j=1}^{N}\frac{2\mathrm{Imp}_j^\infty}{|x+p_j^\infty-\frac{t}{\mathrm{Im}p_j^\infty}|^2}+\delta(t,x)\ ,$$

  where $\delta(t,.)$ scatters as $t\to\infty$.
- Scattering for $(CMDNLS)_-$ or $(CMDNLS)_+$ with mass $<2\pi$?
- Understand the blow up mechanism for $(CMDNLS)_+$.

4. THE SMALL DISPERSION LIMIT

The problem

Consider the Benjamin–Ono equation with small dispersion parameter ε,

$$(BO)_\varepsilon \qquad \partial_t u^\varepsilon + \partial_x((u^\varepsilon)^2) = \varepsilon \partial_x |D_x| u^\varepsilon \ , \ u^\varepsilon(0,x) = u_0^\varepsilon(x) \in H^2(\mathbb{R}) \ .$$

where $u_0^\varepsilon \to u_0$ strongly in L^2, $\sup_{\varepsilon>0} \|u_0^\varepsilon\|_{L^\infty} < +\infty$.
By the L^2 conservation law, for every $t \in \mathbb{R}$ $u_\varepsilon(t,.)$ is a bounded family of $L^2(\mathbb{R})$ as $\varepsilon \to 0$.
What happens to $u_\varepsilon(t,.)$ as $\varepsilon \to 0$?
Formally, one gets the Burgers equation

$$\partial_t u + \partial_x(u^2) = 0 \ , \ u(0,x) = u_0(x)$$

which displays finite time singularities due to crossing of characteristics,

$$u(t,x) = u_0(y) \ , \ y + 2tu_0(y) = x \ .$$

Creates strong oscillations on $u^\varepsilon(t,x)$.

Step 1: Hardy representation of the zero dispersion limit

Theorem

For every $t \in \mathbb{R}$, the solution $u^\varepsilon(t,.)$ of $(BO)_\varepsilon$ converges weakly in $L^2(\mathbb{R})$ to a function $ZD[u_0](t,.)$, characterized by

$$\forall x \in \mathbb{R}\ ,\ ZD[u_0](t,x) = \Pi_+ ZD[u_0](t,x) + \overline{\Pi ZD[u_0](t,x)}\ ,$$

and

$$\forall z \in \mathbb{C}_+\ ,\ \Pi ZD[u_0](t,z) = \frac{1}{2i\pi} I_+((X^* + 2tT_{u_0} - z\mathrm{Id})^{-1}\Pi u_0)\ . \qquad (*)$$

Proof, I

By the L^2 conservation law, $\forall t \in \mathbb{R}\ ,\ \|u^\varepsilon(t,.)\|_{L^2} = \|u_0^\varepsilon\|_{L^2}\ .$
hence $u^\varepsilon(t,.)$ has weak limits as $\varepsilon \to 0$.
Claim : there is only one such weak limit w_t.
Since u^ε is real valued, so is w_t, hence $w_t = \Pi w_t + \overline{\Pi w_t}$ on the real line.
To be proved :

$$\forall z \in \mathbb{C}_+\ ,\ \Pi w_t(z) = \frac{1}{2i\pi} I_+((X^* + 2tT_{u_0} - z\mathrm{Id})^{-1}\Pi u_0)\ . \qquad (*)$$

Proof,II

Explicit formula + elementary scaling argument :

$$\Pi u^\varepsilon(t,z) = \frac{1}{2i\pi} I_+((X^* + 2t\mathrm{e}^{-i\varepsilon t\partial_x^2} T_{u_0^\varepsilon} \mathrm{e}^{i\varepsilon t\partial_x^2} - z\mathrm{Id})^{-1} \mathrm{e}^{-i\varepsilon t\partial_x^2} \Pi u_0^\varepsilon) \ .$$

- iX^* maximally accretive,
- $\mathrm{e}^{\pm i\varepsilon t\partial_x^2} \to \mathrm{Id}$ in $\mathscr{L}(L^2_+(\mathbb{R}))$ for the strong topology of operators,
- $T_{u_0^\varepsilon} \to T_{u_0}$ in $\mathscr{L}(L^2_+(\mathbb{R}))$ for the strong topology of operators,

so

$$g_z^\varepsilon := (X^* + 2t\mathrm{e}^{-i\varepsilon t\partial_x^2} T_{u_0^\varepsilon} \mathrm{e}^{i\varepsilon t\partial_x^2} - z\mathrm{Id})^{-1} \mathrm{e}^{-i\varepsilon t\partial_x^2} \Pi u_0^\varepsilon$$

is strongly convergent to

$$g_z^0 := (X^* + 2tT_{u_0} - z\mathrm{Id})^{-1} \Pi u_0$$

in $L^2_+(\mathbb{R})$, and even in $\mathrm{Dom}(X^*)$ endowed with the graph norm. Consequently, $I_+(g_z^\varepsilon) \to I_+(g_z^0)$. Since $f \mapsto f(z)$ is a continuous linear form on $L^2_+(\mathbb{R})$, formula $(*)$ follows.

Remarks

- *Continuity in time.* The function $t \mapsto ZD[u_0](t,.) \in L^2(\mathbb{R})$ is continuous for the weak topology of $L^2(\mathbb{R})$, and the weak convergence of $u^\varepsilon(t,.)$ to $ZD(u_0](t,.)$ is locally uniform in time.
- *Continuity with respect to the initial datum.* Assume u_0^n converges strongly to u_0 in $L^2(\mathbb{R})$ with the additional condition

$$\sup_n \|u_0^n\|_{L^\infty} < +\infty \ .$$

Then, for every $t \in \mathbb{R}$, $ZD[u_0^n](t,.)$ converges weakly in $L^2(\mathbb{R})$ to the function $ZD[u_0](t,.)$ characterized by formula $(*)$.
Furthermore, the convergence is uniform for t in any compact subset of $\mathbb{R}$.

Step 2 : towards a geometric formula

Assume moreover that u_0 is a C^1 function, tending to 0 at infinity as well as its derivative. On a time interval J containing $t = 0$ and on which it is C^1, the solution u of the Burgers equation with the initial datum u_0 can be described by the method of characteristics.
Indeed, the Burgers equation precisely means that u is constant on every characteristic $(x(t), t)$ defined by $\dot{x}(t) = 2u(t, x(t)), x(0) = y$, which turns out to be the segment $x = y + 2tu_0(y), t \in J$. Therefore $\forall t \in J \ , \ u(t, x) = u_0(y(t, x)) \ ,$ where $y(t, x)$ is the unique solution of the equation

$$y + 2tu_0(y) = x \ . \qquad (char)$$

For longer times t, there may be several solutions of equation $(char)$, which creates singularities.

The crossing of characteristics for the Burgers equation

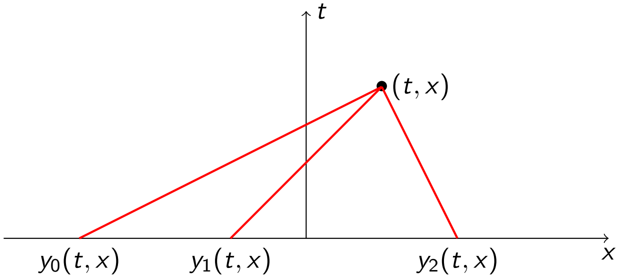

Figure: The crossing of characteristics

For every $t \in \mathbb{R}$, let us denote by $K_t(u_0)$ the set of of critical values of the function

$$f_t : y \in \mathbb{R} \mapsto y + 2tu_0(y) \in \mathbb{R} \ .$$

By the Sard theorem, the set $K_t(u_0)$ is a compact subset of $\mathbb{R}$ of Lebesgue measure 0.
If x belongs to the complement $K_t(u_0)^c$ of $K_t(u_0)$ in $\mathbb{R}$, equation

$$f_t(y) = x \qquad (char)$$

admits a finite number of solutions $y_0(t,x) < y_1(t,x) < \dots$, and the sign of the derivative $1 + u_0'(y_k(t,x))$ must be alternatively positive or negative. In view of the behaviour of $y + 2tu_0(y)$ at infinity, we conclude that this sign must be $(-1)^k$, and that the number of such solutions must be odd. Let us denote it by $2\ell_t(x) + 1$. Of course the number $\ell_t(x)$ is constant if x stays in a connected component of $K_t(u_0)^c$.
It turns out that the values of u_0 at $y_0(t,x), \dots, y_{2\ell_t(x)}(t,x)$ completely characterises the zero dispersion limit $ZD[u_0](t,x)$.

The zero dispersion limit geometric formula

Theorem (PG, 2023)

Assume $u_0 \in L^2(\mathbb{R}) \cap C^1(\mathbb{R})$ *with* $|u_0(x)| + |u_0'(x)| \to 0$ *as* $x \to \infty$. *Then*

$$ZD[u_0](t,x) = \sum_{k=0}^{2\ell_t(x)} (-1)^k u_0(y_k(t,x)) \ , \tag{ZD}$$

where $y_0(t,x) < \cdots < y_{2\ell_t(x)}(t,x)$ *are the real solutions of the Burgers characteristics equation* $y + 2tu_0(y) = x$.

Special cases of potentials : Miller–Xu (2011), Miller–Wetzel (2016) by inverse scattering.
On the circle : L. Gassot (2022, 2023).

The zero dispersion limit geometric formula

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Special cases of potentials : Miller–Xu (2011), Miller–Wetzel (2016) by inverse scattering.
On the circle : L. Gassot (2022, 2023).
Here we will establish formula (ZD) in general, directly from formula $(*)$.

Relation to a formula by Brenier

Formula (ZD) is precisely the transport collapse scheme introduced by Y.Brenier (1987) in order to compute the entropic solution of the Burgers equation according to Kruzhkov. More precisely, if we set $T(t)[u_0] := ZD[u_0](t,.)$, Brenier proved that the entropic solution u is given by the following Trotter formula,

$$u(t,.) = \lim_{n\to\infty} \left[T\left(\frac{t}{n}\right)\right]^n [u_0] \ .$$

That this transport collapse formula precisely gives the zero dispersion limit of the Benjamin–Ono equation is a surprising fact. Furthermore, the fact that $ZD[u_0](t,x)$ does not always coincide with $u(t,x)$ implies that $T(t)$ is certainly not a semigroup.

Proof, I

It is enough to prove formula (ZD) if u_0 is a rational function with simple poles,

$$u_0(y) = \frac{P_0(y)}{Q_0(y)} = \sum_{j=1}^{N} \left(\frac{c_j}{y - p_j} + \frac{\overline{c}_j}{y - \overline{p}_j} \right) \ , \ \mathrm{Im}(p_j) > 0 \ .$$

We shall transform formula $(*)$ for such a special datum u_0. In this case, the set $K_t(u_0)$ is finite, and, for $t \neq 0$, equation $(char)$ is a polynomial equation of degree $2N+1$. If x is a real number in the complement of $K_t(u_0)$, denote by $y_k(t,x), k = 0, \dots, 2N$, the solutions labelled as follows,

$$y_0(t,x) < \cdots < y_{2\ell_t(x)}(t,x) \ ,$$
$$y_{2p-1}(t,x) = \overline{y_{2p}(t,x)} \ , \ \mathrm{Im} y_{2p}(t,x) > 0 \ , \ \ell_t(x) + 1 \le p \le N \ .$$

The implicit function theorem shows that, for $k = 0, \dots, 2\ell_t(x)$, the functions $y_k(t, .)$ are analytic near x, and, by the Cauchy–Riemann equations, for $z = x$, we have

$$\frac{\partial \mathrm{Im}(y_k(t,z))}{\partial \mathrm{Im}(z)} = \frac{\partial \mathrm{Re}(y_k(t,z))}{\partial \mathrm{Re}(z)} = \frac{1}{1 + 2tu_0'(y_k(t,x))} ,$$

which has the sign of $(-1)^k$. If x is shifted into the upper half plane to a complex number z with a small positive imaginary part, we infer

$$\mathrm{Im}(y_{2k}(t,z)) > 0 \ , \ \mathrm{Im}(y_{2p-1}(t,z)) < 0 \ , \ k = 0, \dots, N, \ p = 1, \dots, N \ .$$

Proof, II

For such a complex number z, let us consider

$$g_{t,z} := (X^* + 2tT_{u_0} - z\mathrm{Id})^{-1}\Pi_+ u_0 \ .$$

We have, for every $g \in L^2_+(\mathbb{R})$,

$$T_{u_0}g(y) = u_0(y)g(y) - \sum_{j=1}^N \frac{c_j g(p_j)}{y - p_j} \ , \ \Pi u_0(y) = u_0(y) - \sum_{j=1}^N \frac{c_j}{y - p_j} \ .$$

and we conclude that

$$(y + 2tu_0(y) - z)g_{t,z}(y) = u_0(y) + \lambda(t,z) + \sum_{j=1}^N \frac{\mu_j(t,z)}{y - p_j} \ ,$$

with

$$\lambda(t,z) = -\frac{1}{2i\pi} I(g_{t,z}) = -\Pi ZD[u_0](t,z) \ , \ \mu_j(t,z) = (2tg_{t,z}(p_j) - 1)c_j \ .$$

Proof, III

Since $g_{t,z}$ must be holomorphic in the upper half plane, the right hand side must cancel if $y = y_{2k}(t,z), k = 0, \ldots, N$. This provides a linear system of $N + 1$ equations for the $N + 1$ unknown

$\lambda(t,z), \ldots, \mu_1(t,z), \ldots, \mu_N(t,z)$, from which we infer

Proof, III

Since $g_{t,z}$ must be holomorphic in the upper half plane, the right hand side must cancel if $y = y_{2k}(t,z), k = 0, \ldots, N$. This provides a linear system of $N+1$ equations for the $N+1$ unknown $\lambda(t,z), \ldots, \mu_1(t,z), \ldots, \mu_N(t,z)$, from which we infer $\lambda(t,z) = \dfrac{N(t,z)}{D(t,z)}$ with

$$D := \begin{vmatrix} 1 & \frac{1}{y_0 - p_1} & . & . & \frac{1}{y_0 - p_N} \\ 1 & \frac{1}{y_2 - p_1} & . & . & \frac{1}{y_2 - p_N} \\ . & . & . & . & . \\ 1 & \frac{1}{y_{2N} - p_1} & . & . & \frac{1}{y_{2N} - p_N} \end{vmatrix},$$

$$N := \frac{1}{2t} \begin{vmatrix} y_0 - z & \frac{1}{y_0 - p_1} & . & . & \frac{1}{y_0 - p_N} \\ y_2 - z & \frac{1}{y_2 - p_1} & . & . & \frac{1}{y_2 - p_N} \\ . & . & . & . & . \\ y_{2N} - z & \frac{1}{y_{2N} - p_1} & . & . & \frac{1}{y_{2N} - p_N} \end{vmatrix}.$$

A lemma

It turns out that the above quotient of determinants can be calculated explicitly.

Lemma

Given complex numbers $z_0, \ldots, z_N, p_1, \ldots, p_N$ pairwise distinct, we have

$$\frac{\begin{vmatrix} z_0 & \frac{1}{z_0-p_1} & \cdot & \cdot & \frac{1}{z_0-p_N} \\ z_1 & \frac{1}{z_1-p_1} & \cdot & \cdot & \frac{1}{z_1-p_N} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ z_N & \frac{1}{z_N-p_1} & \cdot & \cdot & \frac{1}{z_N-p_N} \end{vmatrix}}{\begin{vmatrix} 1 & \frac{1}{z_0-p_1} & \cdot & \cdot & \frac{1}{z_0-p_N} \\ 1 & \frac{1}{z_1-p_1} & \cdot & \cdot & \frac{1}{z_1-p_N} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \frac{1}{z_N-p_1} & \cdot & \cdot & \frac{1}{z_N-p_N} \end{vmatrix}} = \sum_{\alpha=0}^{N} z_\alpha - \sum_{j=1}^{N} p_j \ .$$

Proof, IV

From the lemma, we infer

$$\lambda(t,z) = \frac{1}{2t}\left(\sum_{k=0}^{N} y_{2k}(t,z) - \sum_{j=1}^{N} p_j - z\right) .$$

In view of the algebraic equation

$$yQ_0(y) - zQ_0(y) + 2tP_0(y) = 0 \ , \ Q_0(y) = \prod_{j=1}^{N}(y-p_j)(y-\overline{p}_j) \ ,$$

we have the following relationship between the coefficient of $-y^{2N}$ and the roots,

$$z + \sum_{j=1}^{N}(p_j + \overline{p}_j) = \sum_{\alpha=0}^{2N} y_\alpha(t,z) \ .$$

Proof, V

Now we make z tend to x on the real line, so that $y_k(t,x)$ is real for $k = 0, 1, \ldots, 2\ell_t(x)$, and $y_{2p-1}(t,x) = \overline{y_{2p}(t,x)}$ if $p = \ell_t(x) + 1, \ldots, N$.

$$\begin{aligned} ZD[u_0](t,x) &= -(\lambda(t,x) + \overline{\lambda(t,x)}) \\ &= \frac{1}{2t}\left(2x + \sum_{j=1}^{N}(p_j + \overline{p}_j) - 2\sum_{\alpha=0}^{\ell_t(x)} y_{2\alpha} - \sum_{\beta=2\ell_t(x)+1}^{2N} y_\beta\right) , \\ &= \frac{1}{2t}\left(\sum_{\gamma=1}^{\ell_t(x)}(y_{2\gamma-1}(t,x) - x) - \sum_{\alpha=0}^{\ell_t(x)}(y_{2\alpha}(t,x) - x)\right) , \\ &= \sum_{k=0}^{2\ell_t(x)} (-1)^k u_0(y_k(t,x)) . \end{aligned}$$

This is precisely formula (ZD) ! □

Proof of the lemma

Using the formula for the Cauchy determinants, we have

$$A := \begin{vmatrix} z_0 & \frac{1}{z_0-p_1} & \cdot & \cdot & \frac{1}{z_0-p_N} \\ z_1 & \frac{1}{z_1-p_1} & \cdot & \cdot & \frac{1}{z_1-p_N} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ z_N & \frac{1}{z_N-p_1} & \cdot & \cdot & \frac{1}{z_N-p_N} \end{vmatrix} = \sum_{\alpha=0}^{N} (-1)^\alpha z_\alpha D_\alpha$$

$$B := \begin{vmatrix} 1 & \frac{1}{z_0-p_1} & \cdot & \cdot & \frac{1}{z_0-p_N} \\ 1 & \frac{1}{z_1-p_1} & \cdot & \cdot & \frac{1}{z_1-p_N} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \frac{1}{z_N-p_1} & \cdot & \cdot & \frac{1}{z_N-p_N} \end{vmatrix} = \sum_{\alpha=0}^{N} (-1)^\alpha D_\alpha$$

with

$$D_\alpha := \prod_{\substack{\beta\neq\alpha,\gamma\neq\alpha \\ \beta<\gamma}} (z_\beta - z_\gamma) \prod_j (z_\alpha - p_j)\Delta \,,$$

$$\Delta := \frac{\prod_{j<k}(p_k - p_j)}{\prod_{\beta,j}(z_\beta - p_j)} \,.$$

Consequently, we are led to evaluate the following quotient of Vandermonde determinants,

$$\frac{A}{B} = \frac{V(R)}{V(Q)} \ , \ R(z) := zQ(z) \ , \ Q(z) := \prod_{j=1}^{N}(z - p_j) \ ,$$

$$V(P) := \begin{vmatrix} P(z_0) & 1 & z_0 & . & z_0^{N-1} \\ P(z_1) & 1 & z_1 & . & z_1^{N-1} \\ . & . & . & . & . \\ P(z_N) & 1 & z_N & . & z_N^{N-1} \end{vmatrix} .$$

Notice that the linear form V cancels on polynomials of degree at most $N-1$, and on the polynomial $\tilde{P}$ defined as $\tilde{P}(z) := \prod_{\alpha=0}^{N}(z - z_\alpha)$. The lemma then follows from the identity

$$\begin{aligned} R(z) - \tilde{P}(z) &= zQ(z) - \tilde{P}(z) = \left(\sum_{\alpha=0}^{N} z_\alpha - \sum_{j=1}^{N} p_j\right) z^N + \mathbb{C}_{\leq N-1}[z] \\ &= \left(\sum_{\alpha=0}^{N} z_\alpha - \sum_{j=1}^{N} p_j\right) Q(z) + \mathbb{C}_{\leq N-1}[z] \ . \end{aligned}$$

Further properties of the zero dispersion limit

From formula (ZD) and the observation that the sequence $u_0(y_k(t,x)), k = 0, \dots, \ell_t(x)$, is monotonic, we get the maximum principle

$$\inf u_0 \leq \operatorname{ess\,inf} ZD[u_0](t,.) \leq \operatorname{ess\,sup} ZD[u_0](t,.) \leq \sup u_0 \ .$$

In view of the continuity property with respect to u_0, this maximum principle extends to any $u_0 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Another remarkable property is the formula

$$2t\partial_x ZD[u_0](t,x) = 1 - \mu_{2tu_0} \ ,$$

where we have set, for every real valued function $h \in L^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}} \varphi(x) d\mu_h(x) := \int_{\mathbb{R}} \varphi(y + h(y))\, dy \ .$$

Notice that this formula implies some smoothing property : $ZD[u_0](t,.)$ is locally BV on $\mathbb{R}$ for every $t \neq 0$.

Final comments

- Xi Chen (2023) has given a more direct proof of this formula without using the approximation by rational functions. He also relaxed the L^∞ assumption on u_0.
- Rana Badreddine (2024) has studied the similar problem for $(CMDNLS)_\sigma$. She obtained a different formula, which leads to

$$\log |ZD[u_0](t,x)|^2 = \sum_{k=0}^{\ell_t(x)} (-1)^k \log |u_0(y_k(t,x)|^2 \ , \ x \in K_{-t\sigma}(|u_0|^2)^c \ ,$$

where $y_0(t,x) < \cdots < y_{\ell_t(x)}(t,x)$ are the real solutions of $y - 2t\sigma|u_0(y)|^2 = x$.

- Work in progress with E. Blackstone, L. Gassot, P. Miller. Oscillation profile in the above asymptotics (Whitham).

$$\begin{aligned} u^\varepsilon(t,x) &= a(t,x) + b(t,x) Q_{r(t,x)}\left(\frac{\theta(t,x)}{\varepsilon} + \varphi(t,x)\right) + o(1) \ , \\ Q_r(\alpha) &:= \frac{1-r^2}{1-2r\cos\alpha + r^2} \ . \end{aligned}$$

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